VOICES IN THE DARK: MAKING SENSE OF TALK IN MATHEMATICS CLASSES

Gary Ernest DAVIS, Arthur JONES, & Catherine PEARN
La Trobe University, Melbourne

When we talk to each other and to students about mathematics, how do we know we are communicating and making sense, and that we are not simply lonely voices talking in the dark? Communication by means of talk begins in the everyday world of children and adults. Communication in the everyday world can be difficult. However mathematics at school and university level is not an everyday activity for most students and it is conceivable, if not likely, that students and their instructors are talking in parallel, or at cross-purposes, apparently communicating about the same topicbut in reality simply making pronouncements on their own understanding. von Glasersfeld (1990) has said:

> "If it is the case that such conceptual schemas - and indeed concepts in general - cannot be conveyed or transported from one to the other by words of the language, this raises the question of how language users acquire them. The only viable answer seems to be that they must abstract them from their own experience." (p. 35)
>
> "... language is not a means of transporting conceptual structures from teacher to student, but rather a means of interacting that allows the teacher here and there to constrain and thus to guide the cognitive construction of the student." (p.37)

In a 3rd year chaos course taught in 1993 to mathematics students at La Trobe University, students were expected and encouraged to make presentations of their attempts at problem solutions to the rest of the class. They were encouraged to write on an overhead projector and to talk aloud to their solution or attempted solution. These presentations formed an integral part of the course, because it is through them that student dialogue took a questioning, answering, and critical role. These were the times when awareness of a student's own actions become possible, both for the presenting student and other students in the class. In this sense the presentations both heightened a mathematical experience of the presenting student, and made that experience public. It is through student dialogue in these situations that mathematical knowledge, beyond individual knowledge, is co-constructed.

One of the major reasons for our using student presentations is that it gives students an opportunity to give longer, more detailed explanations than they might otherwise give. In this regard Webb (1991) says:

> "The majority of the partial correlations between giving content-related explanations and achievement are positive and statistically significant. Even controlling for ability, giving content-related explanations was

positively related to achievement. This suggests that giving elaborated explanations may be beneficial for achievement." (p. 372)

We agree with von Glasersfeld that concepts are abstracted from personal experience, and it is almost a corollary of this point of view that the use of a phrase such as "the derivative is zero" will mean somewhat different things, conceptually, to different people, and will be highly dependent on their previous experience. For example, an experienced mathematician might read various things into the phrase "the derivative is zero": that the degree one term in the Taylor series at that point is missing; that the tangent vector field at that point is horizontal; that if one knew a formula for the function then one could find an equation for the point at which the derivative is zero. It seems to me that, despite the use of a common language conceptual structures often do not find a common meeting place.

COMMON LANGUAGE, DIFFERENT SCHEMAS

Two students in the chaos course, Shaun and Vaughan, seem to be using a common language: they are discussing the successive iterates of an interval under a tent map $T:[0,1]\rightarrow[0,1]$ defined by $T(x) = 1 - |1 - 2x|$. They use a common language for intervals and iterates, and draw the graph of the tent map in similar ways, and begin using it to illustrate graphical iteration. However, we believe that Shaun and Vaughan have quite different conceptions of how this iteration of an interval is taking place. Shaun appears to follow the end-points of the interval and "plug them" into the formula. Vaughan on the other hand sees the tent map, for the purposes of iterating the interval, as a doubling map. He even goes so far as to relate it to linear expansions treated earlier in the course.

Shaun draws the graph of the tent map and tries to work out the image of the interval [0.4, 0.6] by indicating what's happening on the diagram. He apologises for not being able to draw the image well, and then resorts to applying the formula for f to the endpoints of the interval:

> **Shaun: "That can be better worked out rather than the map by just looking at the formula which is just ...."**

He then attempted to find the image of this interval under the second iterate, but he did not draw the graph of the second iterate.

> **Shaun: " Unfortunately you can't see it off the graph, but if you just, umm, plug the values in you'll find, I think, that's what you'll get."**

Vaughan present his solution to the image of [0, 0.4] under the second iterate, and relates it to a linear map discussed in a previous class:

Vaughan: "After two iterations we've got the interval 0 to 0.4, and using Jack's (the teacher) whatever it was from a few weeks ago, the .. anything from 0 to 0.4 under successive iterations for a while is going to get bigger because it's an example like we had before of the linear.... (waves hands)"

Shaun asks how Vaughan knows the interval is going to get bigger, and Vaughan essentially answers that the tent map is just multiplication by 2 on the left half of the interval:

Shaun: "What do you mean it's going to get bigger?"

Vaughan: " Well, any, any point between 0 and 0.4 under the next iteration will grow larger. Because, basically if you just look at that half (tent map on [0, 0.5]), that half of .. the thing, it's just one of those standard, ah .. .thingies, so basically the point you've gone back to will actually double, so that, ah, interval's going to keep doubling until it hits, ... goes from 0 to 1.

Shaun then asks why the interval will eventually iterate to [0, 1] and Vaughan gives a general, geometric, answer:

Shaun: " How do you know it's going to get to 1?"

Vaughan: "Because when it, when..., when the interval goes more than half of it it's going to start at 0, it's going to always start at 0, eventually it's going to go, well .., in point of fact the next one's going to go from 0 to 0.8, but I mean if you just want to talk about it generally it's eventually going to go past half way."

Shaun: "Won't it go from 1 to 0.8, the next one?"

Vaughan: "No! The next one will go from 0 to 0.8. If you're doing an interval from 0 to 0.4, 0's going to, ah, the iteration's going to send 0 back to 0. It's just going to expand the interval. Then once it gets past half way it's going to, the next iteration's going to expand it to the whole of the interval of 1."

Shaun appeared to have difficulty seeing a geometric interpretation of iteration of the interval under the tent map, and relied instead on substituting the end-point values of the intervals into a formula. The fact that the formula for the successive iterates gets rapidly more complicated may have caused him cognitive difficulty. Vaughan, on the other hand, viewed the problem geometrically from the beginning, and realised that the only formula he needed to solve the problem was multiplication by 2. Vaughan seems to have interiorized the iterative behaviour of linear maps, but there is no evidence from this excerpt that Shaun has done likewise. Shaun's relevant repeatable action scheme in this setting is putting numbers into a formula and computing. This is probably an interiorized and generalized action scheme for him: unfortunately it leads to a complicated tangle in trying to apply it directly, the way he did, to this problem.

It seems that despite their talking about a common problem, and using common language, Shaun and Vaughan are not communicating in a conceptual sense. How, wothout a shared experience, could they?

TELLING

"Telling" can be problematic for a committed constructivist teacher. How does one reconcile a belief that students actively build their mathematics through personal action and reflection with a lecture presentation in which a relatively expert mathematician "tells" about a part of mathematics? The reconciliation for us comes from our understanding of teaching and learning as evolutionary processes: there is nothing intrinsically wrong or right about telling, it simply creates certain environments which may be helpful or inimical to a student's understanding. A constructivist approach to mathematics learning should not be confused with discovery learning. For us, there is nothing in constructivist theory that says students can or will "discover" mathematics for themselves. Indeed, as Steffe has pointed out on numerous occasions, there is no necessity, nor even a likelihood, that mathematical activity will develop out of more everyday activities. And Piaget (1972) has remarked:

> "When one thinks of the many centuries it took to arrive at our mathematics, it would be absurd to think that without the guidance (of teachers and parents) the child could arrive at a clear formulation of the central questions on his own." (p. 21)

We see a constructivist teacher's role as creating those problematic environments that promote mathematical activity in students. These environments are necessarily built by a teacher out of their own mathematical experience and may, but do not necessarily, include careful observation of the actions and inter-personal dialogue of students in mathematics classes. Again we emphasise, each teacher builds their own constructivist understanding and so we, in the teaching team for the chaos course, will have individual understandings of the importance of "telling" at any particular moment. A critical issue however is that we become of those times we "steal" the possibility of mathematical action and reflection for a student by telling them an answer or procedure when they are still engaged in trying to solve a problem, and take steps to avoid such "cognitive theft". A basic problem of constructivist mathematics teaching is the following:

If a teacher pays little or no heed to students' ways of acting mathematically then students won't engage with the teacher's concerns. On the other hand, students are initially unaware of the nature of their action schemes - helping to make them aware is partly a problem of social discourse. This is the problem of constructively reconciling individual and cultural ways of acting and knowing. A really difficult issue for us is: just when is it that well-intentioned critical comments from a teacher constitute an infringement on the thought processes and clarifications of a student? Webb (1991) remarks that:

"... being timely and understandable does not necessarily ensure that explanations will be beneficial to the recipient." (p. 369).

A delicate example of this dilemma occurs in the following interaction. A student, Jim, is explaining to the class his solution to the problem: "Guess the general shape of the graph of f^n (the n^{th} iterate of f) when n gets large, where $f:[0,1]\rightarrow[0,1]$ is defined by $f(x) = x(1-x)$." He draws the graph of the function f and the identity function on the interval $[0, \frac{1}{2}]$, and says that f(x) increases from 0 to $\frac{1}{4}$. He then draws what appears to be the same graph, and says "as x increases from 0 to $\frac{1}{2}$, $f^2(x)$ increases from 0 to $\frac{1}{4}$."

Anna (another student): " Are you sure?"

Jim: "Am I sure?"

Anna: " f(x) as x goes from 0 .... (inaudible)"

Jim: " Oh, sorry! And then ..."

Jim then writes "and then back to 0."

Anna: " If you put one, one second to f squared it gives you ... $\frac{3}{16}$ not $\frac{1}{4}$. So you just draw a graph f, ... and it's not f squared, it's just f."

Jim: "The first one?"

Anna: "No. The second one."

Jim: "Oh, right. I see what you mean."

Anna: "As f(x) increases from ...."

Jim: " Oh, yeah, that's right. Oh yeah, oh sorry. So that should be $\frac{1}{4}$ there and that should be $\frac{1}{2}$."

Jim draws a graph increasing from 0 at x = 0, to $\frac{1}{4}$ at x = $\frac{1}{4}$ and then decreasing to 0 at x = $\frac{1}{2}$. This is *not* the graph of the second iterate of f.

Teacher: "As x increases from 0 to $\frac{1}{2}$, f(x) increases from 0 to $\frac{1}{4}$. Now if f(x) increases from 0 to $\frac{1}{4}$, f of x (*sic*) increases from 0 to f of $\frac{1}{4}$.

Jim: "Yeah, but isn't that $f(\frac{1}{16})$ ... a $\frac{1}{16}$?"

Teacher: " Whatever, yeah. But it's not going back.

Jim: " $\frac{1}{2}$ times $\frac{1}{4}$ times ..."

Teacher: "It just comes back. $\frac{1}{2}$ there (*pointing to the point at which his curve attains a maximum*) and some other value, less than $\frac{1}{4}$ (*pointing to the maximum value of f^2*).

Jim: " $\frac{3}{16}$. No, that's not $\frac{1}{4}$ there."

He erases the maximum value of $\frac{1}{4}$ on the vertical axis of his graph of f^2.

A further discussion ensues with another teacher in the class as to the range of the variable on the horizontal axis of the graph of f^2. The issue, as to whether there is agreement that this is or is not a graph of the second iterate of f, is not resolved.

Apart from the important issue of two people - Jim and the teacher - apparently talking at but not listening to, each other, this interaction raises the point of whether the teacher's telling at that point was beneficial to the student. We believe, *in retrospect*, that it was not and that a continuing dialogue between students *may* have resolved the issue for Jim.

DEFINITIONS

The concepts we discuss here are sophisticated and relatively far removed from everyday experience. At this level of mathematics, definitions are critically important because they are organizing principles, for the teacher, for a variety of phenomena, and because they potentially distinguish, in a literal sense, quite subtly different examples. On the other hand they create a serious problem for students , because they are often used by teachers as the basis for the formation of concept images when a student might not have the appropriate examples to abstract from. Generally, the tenor of statements in the literature concerning definitions has emphasised their negative aspect. For example, Skemp (1982) and Steffe (1990) say, respectively:

> "Concepts of a higher order than those which a person already has cannot be communicated ... by a definition, but only by arranging .... to encounter a suitable collection of examples." (p. 32)

> "Providing a definition can be orienting but using it can be very problematic especially if it has not been the result of experiential abstraction." (p. 100)

However Vinner (1991) points out the need for definitions and their constructively constraining role in advanced mathematics:

> " In technical contexts, definitions might have extremely important roles. Not only that they help forming the concept image but they very often have a crucial role in cognitive tasks. They have the potential of saving you from many traps which are set by the concept image. ... Thus, technical contexts impose on students some thought habits which are totally different from those typical to everyday life contexts." (p. 69)

So whilst students may not, and perhaps cannot, learn concept images from definitions, they nevertheless need definitions in an essential way in advanced mathematical settings. At least one of the major concepts in chaos, presented to the students through a definition, has engendered just those difficulties alluded to by Skemp and Steffe. The definition of sensitive dependence given in the lecture notes is as follows:

> A map $f:[0,1]\rightarrow[0,1]$ has sensitive dependence on initial conditions if the following holds for some $\delta > 0$. For each $x \in [0, 1]$ and for each open interval $I \subseteq [0,1]$ containing x, there is a y contained in I and an integer n such that $|f^n(x) - f^n(y)| \geq \delta$.

Both of these definitions, which appear very similar to students, were presented in the lecture notes in class, and in that same session students were set problems that involved these definitions.

In the excerpt below a student Paul has asked the teacher, in a small group, about the meaning of the term "sensitive dependence. The teacher has heard the conversation on the topic in the group and asks Paul to address the whole class:

> Teacher: "Paul has a question"
>
> Paul (addressing the class): "What does sensitive dependence mean? How do you decide if something has sensitive dependence?"
>
> Maria: "..... if the error's getting bigger you have sensitive dependence, if it's still the same.. getting smaller .. If it's still the same ....
>
> Anna: " Sensitive dependence is if you make some small mistake at the beginning, at the end it's just not at all what you want it to be."
>
> Paul: " So what's that in terms of the, the uhm .. (he then looks back through the lecture notes).

Paul's difficulty seems to be that he has neither a concept image nor can he comprehend the definition. Maria and Anna on the other hand appear to have their own concept images for sensitive dependence, but neither of them express these images in a way that Paul can relate to the written definition.

DISCUSSION

Mathematical knowledge that transcends an individual's knowledge is co-constructed in particular contextual settings. Dialogue, in the form of speech, is a basic agent in this process of co-construction of meaning. There are numerous indications over the past decade that social interaction and peer learning play a major role in the learning of mathematics for most students. (Webb, 1982, 1991; Phelps and Damon, 1989; Yackel, Cobb and Wood, 1991; for example). Leder (1993) points out that:

> "Social interactions are... critical for knowledge construction. Making discussion about mathematics learning commonplace and requiring descriptions and defence - if applicable - of one's schemes and methods should not only make learners more reflective but more confident." (p. 12)

It is a basic tenet of constructivist theory and classroom practice that, to use von Glasersfeld's words, ".. language is not a means of transporting conceptual structures from teacher to student...". Yet communication in mathematics classrooms is increasingly being recognised as of major importance. The language issues of verbal communication between students and teachers, teacher telling, and written definitions, are paramount in mathematics classroom communication. These issues have been examined here through examples in tertiary mathematics to elucidate the difficulties inherent in verbal and written communication even in advanced mathematics classes. Language, as von Glasersfeld says is a means of interacting: but, we must ask, interacting about what? As participants in a communicative activity we make inferences about what is meant by cettain words and phrases - the "aboutness" of a dialogue. This can be difficult enough in everyday affairs: in mathematics classrooms it can be extraordinarily difficult, and we must ask to what extent students and teachers are lonely voices talking in the dark? If we believe we have communicated mathematically, by what means can we test that belief?

REFERENCES

Leder, G. C. (1993) Constructivism: Theory for practice? The case of mathematics. *Higher Education Research and Development, 12, 1*, 5-20.

Phelps, E and Damon, W. (1989) Problem solving with equals: Peer collaboration as a context for learning mathematics and spatial concepts. *Journal of Educational Psychology, 81, 4*, 639-646.

Piaget, J. (1972)*Ou và l'éducation?* Paris: UNESCO

Skemp., R. R. (1982) *The Psych ology of Learning Mathematics*. Middlesex: Penguin Books.

Steffe, L.P. (1990) Inconsistencies and cognitive conflict: A constructivist's view. *Focus on Learning Problems in Mathematics*, 12, 99 - 109

Vinner, S. (1991) The role of definitions in the teaching and learning of mathematics. In D. Tall (ed.) *Advanced Mathematical Thinking*, pp. 65 - 81. Dordrecht: Kluwer.

von Glasersfeld, E. (1990) Environment and communication. In L.P. Steffe and T. Wood (eds.) *Transforming Children's Mathematics Education. International Perspectives*. New Jersey: Lawrence Erlbaum.

Webb, N. M. (1982) Group composition, group interaction and achievement in cooperative small groups. *Journal of Educational Psychology, 74*, 475 - 484

Webb, N. M. (1982) Peer interaction and learning in cooperative small groups. Journal of Educational Psychology, 74, 642 - 655

Webb, N. M. (1982) Student interaction and learning in small groups. *Review of Educational Research, 52,* 421 - 445

Webb, N. M. (1991) Task-related v erbal interaction and mathematics learning in small groups. *Journal for Research in Mathematics Education, 22, 5,* 366 - 389.

Yackel, E., Cobb, P. and Wood, T. (1991) Small group interactions as a source of learning opportunities in second-grade mathematics. *Journal for Research in Mathematics Education 22, 5,* 390-408.